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Fractions: conceptual and didactic aspects

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Summary. *In this paper we present the findings of a principally bibliographical long-term research project, concerning "fractions". This is one of the most studied questions in Mathematics Education, since the learning of fractions is one of the major areas of failure. Here we present a way of understanding lack of success based on Mathematics Education studies, rather than on mathematical motivation.*

The teaching-learning process regarding fractions is certainly one of the most studied since the very beginning of research into Mathematics Education, probably because (together with the related question of decimal numbers) it represents one of the most evident areas of failure in schools the world over.

1. Mathematical aspects

It must be said that a number of teachers are unaware of the fact that there is a considerable difference between a fraction and a rational number (this study will deal only with absolute rational numbers Q^a). Few are aware of the purpose of constructing Q^a starting from ordered pairs of $N \times N^+$. The fact that an absolute rational number is a class which contains infinite ordered equivalent pairs of natural numbers (the second of which is not zero) is by no means clear to all.

Let us examine a mathematically acceptable definition of Q^a, starting from N, by considering the pairs $(a; b)$, $(c; d)$ of the set $N \times (N-\{0\})$, where a, b, c, d are any natural number, with the sole restrictions $b \neq 0$, $d \neq 0$, and taking the following relation (indicated by **eq**):

$$[(a; b) \ \underline{\mathbf{eq}} \ (c; d)] \text{ if and only if } [a \times d = c \times b].$$

This relation belongs to a special category, that of the relations of equivalence, in that is [leaving aside the simple demonstration]:

- *reflexive*: for each pair (a; b) of N×(N-{0}), the following statement is true: (a; b) <u>**eq**</u> (a; b);
- *symmetrical*: for each pair of pairs (a; b), (c; d) of the set N×(N-{0}), the following statement is true: ***if*** [(a; b) <u>**eq**</u> (c; d)] ***then*** [(c; d) <u>**eq**</u> (a; b)];
- *transitive*: for each set of three pairs (a; b), (c; d), (e; f) of the set N×(N-{0}), the following statement is true: ***if*** {[(a; b) <u>**eq**</u> (c; d)] ***and*** [(c; d) <u>**eq**</u> (e; f)]} ***then*** [(a; b) <u>**eq**</u> (e; f)].

In this way the initial set N×(N-{0}) can be distributed by subdividing it in equivalence classes via the operation known as "passage to the quotient", thus indicated: [N×(N-{0})] / <u>**eq**</u>.
[N×(N-{0})] / <u>**eq**</u> contains infinite classes, which are the elements that constitute it; in each class there are infinite pairs of natural numbers.
[N×(N-{0})] / <u>**eq**</u> is the set Q^a. Each infinite class of equivalent pairs is called an absolute rational number.
Thus an absolute rational number is a class which contains infinite pairs of equivalent natural numbers; normally a representative for each class is chosen and can be expressed through different written forms.
In Q^a the operation of division can be defined, whereas in N it was not: to divide the pair (a; b) (with $b\neq 0$) by the pair (c;d) (with $d\neq 0$), we need only multiply (a; b) by (d; c) (with the same necessary condition $c \neq 0$).

2. The history of fractions

To better control obstacles, Brousseau has taught us to face the problems that the history of a given discipline poses. We find that the history of fractions, while apparently simple, contains many features of interest.
The history of fractions is long and curious, but we do not have space to discuss it here. Nice examples begin in Egypt from 3,000 B.C., where fractions, principally the unitary ones, were used in sophisticated problems. The word "fraction" comes from the late Latin "fractio", "part obtained by breaking", and thus from the verb "frangere", "to break". Thus we should avoid imagining that the original etymological meaning of the term fractions presupposes that the parts obtained by breaking are "equal".

The symbol $\frac{m}{n}$ is of uncertain origin, but was certainly used by Leonardo Fibonacci Pisano in his *Liber Abaci*, published in 1202. Numbers which are fractions are called "rupti" or "fracti" and the horizontal line traced between numerator e denominator is called "virgula", i.e. "little stick" (from "virga", "stick"). The words "numerator" and "denominator" are also of uncertain origin, but we know that their use became established in Europe during the fifteenth century. The distinction between “proper”, “improper”, and “apparent” fractions dates from the eighteenth century.

The representation of decimal numbers comes from the work of Simone of Bruges, known as Stevin, (1548-1620). He did not, however, use the point, but rather a quite different symbolism: for example, he wrote 34⓪6①5②2③ rather than 34.652.

3. Fractions as an object of scholastic knowledge

The passage from “Knowledge” (academic) to “learned knowledge” (of the student) is the result of a long and delicate path leading first to the knowledge to be taught, then to the knowledge actually taught and finally to the knowledge learnt.
In this sequence the first step of transforming “Knowledge” into “knowledge to teach” is called *didactic transposition* and constitutes a moment of great importance in which the professionalism and creativity of the teacher are of utmost importance.

As we have already seen, the object of Knowledge Q^a cannot simply be transferred to the pupil, neither at primary nor at secondary level. The pupil simply does not possess the critical maturity or cognitive ability to construct such Knowledge.

Nonetheless, among the “learned knowledge”, history, tradition and contemporary society all consider necessary to include Q^a, together with the use of the point, of decimal numbers including those between 0 and 1, and so on. Moreover, the monetary system of almost all countries presupposes that citizens should possess a basic ability to handle absolute rational numbers; the international measurement system adopted it since the end of the eighteenth century, making it necessary; and in practically all jobs it is at least necessary to grasp the intuitive meaning of 0.5 or 2.5. Thus rational numbers have a social statute that makes them an ability that all should develop.

At the same time, it is simply not possible to teach Q^a at both primary and secondary level in a mathematical form which is formally correct.
Thus an act of didactic transposition is clearly necessary in order to transpose Q^a into something accessible to primary and then secondary pupils. The history of Mathematics teaching clearly places the path of this transposition within the following line of development: fractions (primary

and secondary school), decimal numbers (primary and secondary school), Q^a (upper secondary school or, at times, university).

It would be wrong to suppose that "didactic transposition" is the same as "simplification". Often the concepts that our student must go through are bristled with complications, compared to the ones relative to the Knowledge. For example, with fractions numerous conceptual problems arise concerning objects of knowledge which do not exist in Q^a. If we consider the apparent fractions ($\frac{m}{n}$ with n divisor of m) or improper fractions ($\frac{m}{n}$ with $m>n$), their presence is cumbersome and complex, bristled with cognitive difficulties, while in Q^a these options simply do not exist.

Indeed, if it were possible to avoid passing via fractions and go straight to absolute rational numbers things might well be more simple and natural. But this is impossible. It still seems natural to pass via fractions, even if it is not at all clear that this is the most effective path. What is clear is that it poses many difficulties.

Thus fractions, while not a part of academic Knowledge, are nonetheless an issue of Mathematics Education as an object of knowledge, a knowledge that we could call "scholastic".

4. Theoretical framework didactic researches into fractions

Introducing the concept of fractions has a common basis the world over. A given concrete unit is divided into *equal* parts and some of these parts are then taken. This intuitive idea of fraction of the unit is clear and easily grasped, as well as being simple to modelize in everyday life. It is, however, theoretically inadequate for subsequent explanation of the different and multiform interpretations given to the idea of fraction. As we shall see, one single "definition" is not sufficient.

When a child of between 8 and 11 years of age has understood that $\frac{3}{4}$ represents the concrete operation of dividing a certain unit in 4 *equal* parts, of which 3 are then taken, it would seem that everything is proceeding smoothly. Unfortunately, almost immediately it is clear that the simple construction of that knowledge is blocking the way, it is an obstacle to subsequent real learning. This is knowledge, but inadequate to continue in the construction of further correct knowledge. If, for example, we have a unit divided in 4 *equal* parts, what does it mean, from this point of view, taking $\frac{5}{4}$ of it?

At times it seems that many teachers are unaware of the conceptual and cognitive complexity involved. I believe that it is necessary to dedicate a whole section to different ways of intending the concept of fraction, that we would like the pupil to acquire.
To give reliability to my work I am obliged to propose an overview of international research in this delicate field, certainly one of the most cultivated the world over. It is impossible to quote the whole of these researches, since its vastness goes beyond our imagination. I will quote only the works which have been directly influential for my subsequent choices, dropping the others. They will be rather a lot anyway. My hope is that this painstaking bibliographical research (I shall propose mainly quotations with regard to the period 1970-1990, and a more detailed bibliography with reference to the period 1990-2000) may be of use also to others who wish to pursue research in the same field. It has not been trivial constructing it.

4.1. Basic premises

The idea of fractions is formally introduced at primary school level, in Italy usually in the third year, even though it is already present in the most immediate sense of "half" an apple or "a third" of a bar of chocolate, or divide a handful of chocolates in 4 equal parts, at a much earlier age.
What schooling does is formalise the written form and institutionalize its meaning.

Roughly speaking, we can say that the universal first approach is that of taking a "concrete object of reference", considered as unit, which should have the following requirements:
- be perceived as pleasant and thus fun,
- clearly unitary and
- already familiar, thereby not requiring further learning.

Normally a round cake or a pizza is chosen in almost all countries the world over; both these objects have the above requirements

Situations are then imagined in which this given *unit* (a cake, a pizza or similar) must be shared between a number of pupils or people in general. In this way the pupils arrive at the idea of a half (dividing by 2), a third (dividing by 3), and so on: the "Egyptian fractions", which are our first historical example.

For each of these fractions specific written forms are established that for the above cases are $\frac{1}{2}$ and $\frac{1}{3}$ and reading these forms as "a half" and "a third" poses few problems. Nor does generalizing from these examples the written form $\frac{1}{n}$, which assumes the meaning of an initial unitary object divided into *n equal* parts. With young pupils various examples are considered, assigning different appropriate values to *n*.
If then the guests, for different reasons, have the right to different amounts of the equal parts into which the unitary object had been divided, this gives rise to different written forms such as $\frac{2}{5}$ (two fifths) meaning that two of the five *equal* parts into which the unitary object had been divided are taken. Several new ideas thus arise and a number of characteristics of these written forms are then established:

- the number beneath the little horizontal line is called the *denominator* and this indicates the number of *equal* parts into which the unit has been *divided*;
- the number above the little horizontal line is called the *numerator* and this always indicates the number of parts *taken* (in this way, the numerator expresses the number of times the fractional part must be taken, and thus a *multiplication*);
- to give sense to this, the fractional parts of the unit must be *equal*, a point much stressed and to which we will return later in a critical way.

We shall see how the understanding of these elements, and in particular those marked by italics, end up being an obstacle to the construction of the concept of fraction.

4.2. The theoretical framework of didactic researches into fractions

4.2.1. From the 1960s to the 1980s

The years between 1960 and 1980 gave rise to an enormous quantity of studies concerning the learning of fractions by pupils from 8 to 14 years of age, particularly in the United States. These studies principally concentrated on:

- general questions regarding the very concept of fraction (Krich, 1964; Green, 1969; Bohan, 1970; Stenger, 1971; Coburn, 1973; Desjardins, Hetu, 1974; Coxford, Ellerbruch, 1975; Minskaya, 1975; Kieren, 1975, 1976; Muangnapoe, 1975; Williams, 1975; Galloway, 1975; Payne, 1975; Novillis, 1976; Ellerbruck, Payne, 1978; Hesemann, 1979);
- operations between fraction and relative difficulties [Sluser, 1962 (division); Bergen, 1966 (division); Wilson, 1967 (division); Bindwell,

1968 (division); Green, 1969 (multiplication); Coburn, 1973 (addition and subtraction); Streefland, 1978 (subtraction)];

- differing interpretations of the idea of fraction [Steffe, Parr, 1968 (ratio); Coburn, 1973 (ratio) Suydam, 1979 (ratio); Streefland (1979) (measure and ratio)].

Within these pioneering studies, those of Kieren (1975, 1976), that deal with the previous topics, emerge highlighting the existence of at least seven different meanings of the term "fraction", showing how in this polysemy the principal problem of learning is hidden, both concerning the general concept and operations.

During the 1980s the following rich and specific studies appeared:

- learning in general (Owens, 1980; Rouchier et al., 1980; Behr, Post, Silver, Mierkiewicz, 1980; Hasemann, 1981; Behr, Lesh, Post, Silver, 1983; Lancelotti, Bartolini Bussi, 1983; Pothier, Sawada, 1983; Streefland, 1983, 1987; Hunting, 1984a, 1986; Behr, Post, Wachsmuth, 1986; Dickson, Brown, Gibson, 1984; Streefland, 1984a, b, c; Kerslake, 1986; Woodcock, 1986; Figueras, Filloy, Valdemoros, 1987; Hunting, Sharpley, 1988; Behr, Post, 1988; Ohlsson, 1988; Weame, Hiebert, 1988; Centino, 1988; Chevallard, Jullien, 1989);
- learning operations between fractions [Streefland, 1982 (subtraction); Behr, Wachsmuth, Post, 1985 (addition); Peralta, 1989 (addition and multiplication)];
- comparisons between the values of fractions and/or decimal numbers and difficulties in extending natural numbers to fractions and/or decimals [Leonard, Grisvard, 1980; Nesher, Peled, 1986; Resnik et al., 1989];
- problems connected with the differing interpretations of the term "fraction" [Novillis, 1980a, b (positioning on the line of numbers); Ratsimba-Rajohn, 1982 (measure); Hunting, 1984b (equivalence); Wachsmuth, Lesh, Behr, 1985 (ordering); Kieren, Nelson, Smith, 1985 (partition; use of graphic algorithms); Giménez, 1986 (fractions in everyday language; diagrams); Post, Cramer, 1987 (ordering); Wachsmuth, Lesh, Behr, 1985 (ordering); Ohlsson, 1988 (the semantics of the fraction); Davis, 1989 (the general sense of fractions in everyday life); Peralta, 1989 (various graphic representations)].

Particularly significant, amongst the works of the 1980s are the studies of Hart (1980, 1981, 1985, 1988, 1989; with Sinkinson, 1989), although based on many of the previously mentioned studies, he continues in a critical manner mainly the previous works of Kieren and those of the 1980's(Kieren, 1980, 1983, 1988; with Nelson, Smith, 1985). As we shall see, both Hart and Kieren will be active later, too.

I will follow for the mean time this classical stream and I will later present in section 5 the various meanings of fraction, basing my description on the seven classic proposed by Kieren, but also using the work of Hart and the panorama presented by Llinares Ciscar, Sánchez García (1988). I will obviously need to modernize this point of view, to take advantage of the instruments that meanwhile have been developed by Mathematics Education research.

In the early Eighties, more precisely in 1980 and 1981, two articles appeared by Guy Brousseau (1980c, 1981), concerning decimal numbers didactics and based on experiences during the 1970s in the Primary school "J. Michelet" in Talence, France, which are considered a milestone in the field.
Mentioned articles are fundamental for the evolution of Mathematics Education not only for the object investigated but mainly for a new methodology (named at the time "experimental epistemology", completely new in the world panorama), rigorously discussed by the author prior to presenting it to the international community, that we can consider as the starting point of the modern idea of research in Mathematics Education.

In these articles, the author defines the set D of decimal numbers as an extension of N which will then enable the passage to the rational numbers Q, studying briefly its history and its algebraic characteristics. He then demonstrates a highly interesting didactic sequence, now considered historical, that exploits primary school experiences ("repeated ten times", he asserts, before publication of the results). The first step involves using the pantograph to analyze fractions, the second an enlarged reconstruction of a given puzzle, and third a problem concerning different thicknesses of sheets of paper. In each step Brousseau analyses in detail aspects now considered belonging to didactics but which at the time were absolutely new. He then describes the results of a test to assess this activity. The study proposes in ever-increasing depth an approach to rational numbers based on decimals, and constantly analyses each phase of the experimental process.

The scope of the work is such, under a general point of view, that it must be considered a turning point in the structure of Mathematics Education research. The Author himself realized this, and in his later paper about epistemological obstacles (Brousseau, 1983, surely one of the most cited articles by researchers in Mathematics Education) he explicitly remembers as an example his didactic study of decimal numbers.

The school is, however, slow to assimilate and absorb the results of didactic research. Even after 20 years, the effect of this article on school praxis has not been what it could or should have been. Moreover, analysis of classroom practices and textbooks production shows how much is still to be understood concerning the difficulties encountered in introducing fractions and decimals.

Another significant contribution is that of a project conducted in the USA from 1979 to 2002, in which a group of researchers, (K. Cramer and T. Post (University of Minnesota), M.J. Behr (University of Illinois), G. Harel (University of California) e R. Lesh (Purdue University), launched *The Rational Number Project*, giving rise to more then 90 articles up to 2003. The focus of this research is rational numbers, and all that accompanies them in the field of “proportional reasoning”, therefore including explicit and significant reference to fractions.

I have quoted some of these articles and I will have to quote others soon. The Reader can find them in the bibliography, even if they are not recognizable as part of the project but according to the names of the Authors. Exhaustive reference to this research is impossible within the scope of this study, but can be found in the internet site which bears the name of the project; in this website the Reader will found the history of the project and the complete bibliography. both in alphabetical and in chronological order.

4.2.2. From the 1990s to the present day

Within this period the research in the area of fractions, decimal numbers and introduction to rational numbers (with reference to Primary and Secondary School, pupils aged 6-14 years) is voluminous. To avoid dispersing myself in references that I will not use and to avoid a terribly long bibliography, the following is merely a list of those research studies which influenced what I will discuss later.

Clemens, Del Campo (1990) begin a trend pursued in different ways by many researchers. The conceptualization of rational numbers is considered clearly not a natural process, but rather it evolves corresponding to a need of human beings (there is a wide agreement about this idea in international literature The Authors suggest that, as there are discussions on opposite sides, it is better to avoid this debate.

Weame (1990) criticizes the organization of both procedural (often used in school) and conceptual learning, proposing an example concerning the constructing of sense for decimal numbers.
Saenz-Ludlow (1990, 1992, 1994, 1995) initiates a line of development subsequently much followed, that of “case studies” on learning fractions (but Hunting 1986 already published a similar work), i.e. analyses of the teaching-learning process concerning a single subject focusing on personal

strategies employed both for conceptualizing fractions and for spontaneous performing of addition involving fractions.

Davis, Hunting (1990) suggest to carry out parallel didactic activities for the promotion of different skills regarding fractions, for example in discrete and continuous contexts, continuing in the line of the research carried out by Hunting and Korboski during the 1980s.

Mack (1990, 1993) proposes the idea of “informal knowledge” based on every day life spontaneous activities to answer problems concerning individual’s real life, demonstrating through his classroom activities that this informal knowledge can permit an initial construction of ideas concerning fractions and rational numbers in which the former are treated as parts of a whole, with each part considered as a number in itself rather than as a fraction.

Bonotto (1991) presents a detailed analysis of various approaches to rational numbers and related didactic experiments.

Basso (1991a, b, 1992) suggests possible didactic paths for fractions, particularly in the 4th and 5th primary school classes, based on the results of didactic research conducted in the 1980s.

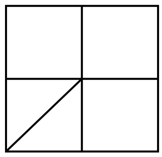
Figueras (1991) presents an interesting and wide summary of the use of fractions and rational numbers in the real world, providing a number of interesting points of departure for didactic examples based on socially concrete situations in the use of fractions.

Hunting, Davis (1991) emphasize the relationship between the idea of ratio and the first learning of fractions, suggesting, from the beginning, a development of the two concepts together, a strategy also proposed by Streefland (1991) and Neuman (1993).

Hunting, Davis, Bigelow (1991), provide a critical analysis of didactic experiences and claim the need of a long-term treatment and consolidation of the unitary fraction before any other step, proceeding then to further “fractioning” of the parts thus obtained, but maintaining constant reference to the original unit. This position is also supported by Kieren (1993a) and Steffe, Olive (1990).

Streefland (1990, 1991, 1993) defends and provides examples of approaches to the teaching-learning of fractions within the real world to justify step by step the needs that derive from daily life as regards the learning and mastering of fractions and rational numbers.

Valdemoros (1992, 1993a, b, c, 1994a, b, 1997, 1998, 2001, et al. 1998) provides a considerable diversity of perspectives on the language of fractions, in particular studying the construction of the meaning of fractions via different symbolic systems and with reference to concrete materials and models. Valdemoros (1997) examines the intuitive resources that can help with addition of fractions, often with reference to single case studies. Valdemoros (2004) demonstrates the results of a research project involving 37 pupils aged 8-11, based on the different contents that can be assigned to a fraction. Particularly interesting is the case of students who give the following answer to the question of how to divide a square wall into five equal parts between five painters:



On various occasions, I have myself found the same answer. The author distinguishes between different levels of data analysis: semantic, syntactic, "translation" from one language to another, the language of arithmetic, of reading.

There are many studies based on classroom experiences concerning pre-school fractions activities. In particular the work of Pepper (1991), Hunting, Pepper, Gibson (1992) will be later mentioned when referring to teaching proposals.

Cannizzaro (1992) analyses the relationship between mathematical, cognitive and curricular levels in the didactics of arithmetic, at a certain point she examines the didactics of fractions emphasizing a number of questions to which we shall return, such as the need to distinguish different acceptations of the term fraction and the risks inherent in the use of concrete models.

Kieren (1992, 1993a, b, c) continues the classic studies of the previous 20 years, insisting on the idea of skills as specific personal facts and proposing and verifying the existence of personal mechanisms for the construction of knowledge in this field, proposing learning processes which begin with the "fractioning" of units.

Behr, Lesh, Post, Silver (1992, 1993) provide a critical discussion of teaching activities that are contemporary to them, distinguishing between stages in the learning of fractions and rational numbers and analyzing the language of fractions didactics in the classroom.

Bonotto (1992) presents the results of a test on fractions and decimal numbers administered with fifth year primary and first year high school pupils. The study focuses in particular on ordering and shows how knowledge of natural numbers is both an aid and an obstacle to this learning, how there are difficulties in handling the passage from fractions to decimal numbers and how knowledge of fractions and of decimals can enter into conflict. There is thus the need for a long-term path of adaptation in the learning of these concepts.

Gray (1993) provides a study of the general problems encountered in the passage from natural numbers to fractions and relative mathematical and learning difficulties.

Davis, Hunting, Pearn (1993a, b) propose the use of diagrams to show the relationship between natural numbers and fractions, after having verified the pupils' ability to move from one to the other. Their study documents a teaching experiment over two school years with pupils of 8-9 and 9-10 years of age.

Ball (1993) presents a personal teaching-learning experience with pupils in the third year of primary school, involving long discussion of everyday uses of fractions in ordinary language, the construction of a solid awareness and the consolidation of personal representations before he discusses the use of symbols, which are openly negotiated.

Bezuk, Bieck (1993) insist on the linguistic mastery of work on fractions trying to give sense to its learning and use; in their book they propose a brief summary of research conducted in this direction.

Graeber, Tanenhaus (1993) propose an informal approach to fractions designed to give them a concrete sense, using fractions as numbers for measuring sizes and thereby they bring students to build an informal knowledge of this topic.

Brown (1993) emphasizes the necessity to find common theories and models in order to overcome the difficulty deriving from the fragmentation of the research into learning difficulties of our subject.

Giménez (1994) proposes a distinction between "divide" in everyday language and "fraction" in Mathematics, using accounts of experiences, stories and various cognitive stimuli together with history and classroom discussion designed to create greater cultural integration between different situations of use of fractions.

Groff (1994) is one of numerous researchers who believe that fractions should be banished from the first years of schooling. Others argue the contrary, sustaining, as we shall see, that informal construction of the idea of fraction should begin in the Infants school. I personally believe that the latter position is correct, while it is clearly necessary to proceed with caution.

Mariotti, Sainati Nello, Sciolis Marino (1995) examine the skills that students say they possess at the moment of passage from Intermediate to High secondary school, discovering, through the following study in depth questions and answers, that they generally believe that different sets of numbers are disjoint and that the written form used determines the nature of the number.

Kamii, Clark (1995) consider the classic question of the difficulty of understanding the relation of equivalence between fractions, I will use the results of this research when I will propose didactic comments on the learning of fractions.

Pitkethly, Hunting (1996) provide an extremely wide and useful critical summary of many of the points of view, trends and directions of research in this field, obviously without explicit didactic suggestions, but only giving a wide panorama of research up to mid 1990s. We hope that soon a similar work will be carried out, starting from the work of Pikethly and Hunting, up to 2005 The present study is an attempt to participate in the construction of a similar summary for the period up to 2005.

Sensevy (1996a) describes a 2 years teaching-learning experience with fourth and fifth year primary pupils concerning the constant negotiation of meanings and construction of formalisms in the classroom while conceiving problems involving fractions. The common and recognized objective was that of achieving effective communication. Meanings and formal models were based on appropriate semiotic tools designed to enable students and teacher to share the involved meanings. Obviously new social norms arise that determine a new form of didactic contract. Sensevy's experience is based on a theoretical study (1994) carried out during his doctoral thesis.

Sensevy (1996b) is also responsible for an activity of Didactic Engineering called "The Journal of Fractions", conducted over a period of two years in fourth and fifth year primary classes. The study focuses on the temporal contracts which weigh upon the teaching-learning process, contracts which impede the pupil from becoming expert because new input is constantly presented before allowing previous content to be fully assimilated and mastered. The didactic experimentation-research, in which the pupils play an active role, was designed «to study the temporal conditions that can bring the pupil build a reflective activity within an epistemological framework» (p. 8). While the research does not directly concern fractions or decimal numbers, but rather general issues, it does however choose an example based on fractions extremely pertinent to our discussion.

Barbero, Carignano, Magnani, Tremoloso (1996) examine concrete data concerning errors which students commit when working with fractions, analysing situations and causes, as do Bonotto (1991, 1993, 1995, 1996), Bonotto, Basso (1994), Bove et al. (1994).

Vaccaro (1998) makes a proposal for didactics of fractions during the final years of primary school through using an appropriate fairy story. It is worth noting that this paper explicitly suggests some activities, so it is a contribution in the field of "didactic engineering".

Zazkis (1998) studies the polysemy in scholastic mathematical practice, a widespread theme in current didactic research, focusing in particular on the ambiguities in the use of the terms "divisor" and "quotient" and the negative effects of this on school practice, classroom language and learning (the text is based on interviews with students). Clearly this polysemy causes difficulties in the learning of fractions.

Hahn (1999) illustrates the results of a research study, apparently disjoint with our topic, into the real skills which underlie the work of the salesman which shows that "the sole mathematical concept that trainee salesmen master, is that of calculating percentages" (p. 229, my translation). The author then studies the same type of skills as possessed by students at different levels. Since the concept of fraction lies at the heart of percentages and of its calculations, I shall later make use of these results. The study concerning the learning of percentages is renowned and dates more than 20 years (for instance Noelting, 1980; Karplus, Pulos, Stage, 1994; Adda, Hahn, 1995).

Weame, Kouba (2000) present a discussion of conceptual difficulties in the learning of rational numbers, as highlighted by an evaluation study of the progress in national education in the USA.

Singh (2000) presents a study of ratio and proportion, an important theme for our purposes, since ratio is one of the possible semantic acceptations of the term fraction. As the author states, activities concerning proportions require the ability to handle simultaneously two ratios. Karplus, Pulos, Stage (1983); Hart (1988); Lamon (1993); Resnick, Singer (1993); Karplus, West (1994); Confrey (1994, 1995) have all made important contributions, concerning the same field of enquiry, established as an independent direction of research.

In the following pages, in order to study the different acceptations of "fraction" and general didactic problems in the learning of fractions, and in order to supply some didactic suggestions, I shall make reference to some results of the following specific works.

Adjiage, Pluvinage (2000) illustrate the results of a test administered in class over a period of 2 years in which the one-dimensional rather than a two-dimensional geometrical representation is used. Their conclusion is that the classical two-dimensional representation causes well known obstacles, difficult to overcome, whereas a constant, balanced use of both can reduce those difficulties. They also underline the importance of making reference to sizes which are common in the everyday lives of pupils, something already emphasized by Carraher, Dias Schliemann (1991).

Keijzer, Terwel (2001) present a interesting "case study" conducted over a period of 30 lessons in a Primary school in Holland. The objective was to construct a series of basic skills in handling fractions, i.e. to teach students fractions literacy skills . The authors describe the process of defining objectives, the single lessons, the constructing of learning and the tests used to assess the skills developed, taking account of recent didactic researches that highlight difficulties in order to prevent and overcome them. The study contains interviews between teachers and learners as well as diagrams and drawing produced by the latter. The study is based on the findings contained in Keijzer, Buys (1996) (which also contains a specific proposal for a curriculum concerning fractions).

O'Connor (2001) presents a discussion group of fifth year Primary school children who consider the question: "Can any fraction be transformed into a decimal number?". The goal of this work is general and the fraction as a mathematical object is chosen as a theme, not as the final objective. The objective is to show how the work of teachers often encounters problems arising from individual personal interpretations on the part of students, due both to mathematical complications and interference with calculations. The study is also particularly useful within the study of fractions for the way in which it sheds light on spontaneous constructions of knowledge.

Llinares (2003) (in Chamorro ed., 2003, Chapter 7) examines Mathematics Education at Primary School, considering the basic elements of Mathematics presented as didactic suggestions to teachers, without neglecting general and specific results of Mathematics Education research. It is a set of suggestions for teaching practice and of reflections on possible learning results It comes out useful to see how Llinares, a protagonist in the study of fractions didactics in the Spanish and Latin American world at the end of the 1980s, considers and uses the results of Didactics of the 1990s. This work will be certainly useful in the last chapter of our paper.

Spain is always active in this kind of operations that give “high” support to the teaching practice. A text published in 2001, with similar objectives, edited by Castro (2001), includes an analogue chapter on fractions, still for primary school (Castro, Torralbo, 2001), one specific on decimal numbers again written by Castro, one on proportion (Fernández, 2001).

In the same direction moves (Socas, 2001) in a chapter on decimal numbers with many didactic concerns that can be found in a university manual for Primary school teaching training.

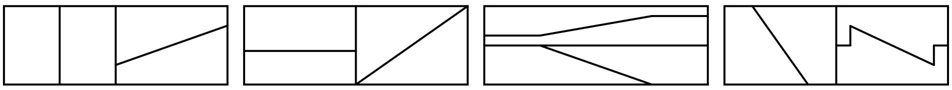
We can found a similar approach in Cid, Godino, Batanero (2003), another book dedicated to didactics focused on numerical systems which contains chapters on fractions and positive rational numbers (Chapter 4, pp. 159-196) and decimal numbers (Chapter 5, pp. 197-232), once again with many didactic issues, not only teaching suggestions but also reflections on learning processes (that become important and concrete suggestions for the teacher). These studies, too, will be very useful to our reflections, later.

Gagatsis (2003) is a collection of Mathematics Education articles, translated into Italian, regarding a research conducted by the Greek researcher in Greece and Cyprus, with a preface by Raymond Duval. Chapter 2 is dedicated to representations and learning, with many examples, and Section 3 (pp. 82-95) to fractions. The author poses the following research question: «Is there a form of representation of the concepts of equivalence and addition between fractions which students tend to handle with greater ease? Is there a mode of translation of representations relative to the concepts of equivalence and addition between fractions which students tend to handle with greater ease?» (pp. 83-84). A research study involving 104 fifth form primary pupils from Cyprus shows students’ limited flexibility in moving from one representation to another and thereby difficulty in choosing an effective representation of addition and equivalence between fractions. In this book one can find references to other researches of the same Author dealing with the issue of fractions and its representations, as Marcou, Gagatsis (2002).

We mentioned a lot of works, and this points out the complexity and the large number of studies devoted to the field. In conclusion, some reference must be made to an important mainstream, which had no analogues nor premises in 1960s and 1970s, but starting from the 1980s it has acquired constant raising importance , i.e. the use of new technologies in didactics in general, in Mathematics Education in specific, in the didactics of fractions, and moreover of decimal and rational numbers. As the field is beyond the scope of this study, here I will limit reference to Chiappini, Pedemonte, Molinari (2004), one of the most recent works dedicated to this area, containing an ample bibliography for those interested in pursuing further research.

5. Different ways of understanding the concept of fraction

Something which often strikes teachers on training courses is how an apparently intuitive definition of fraction can give rise to at least a dozen different interpretations of the term.
1) A fraction as part of a one-whole, at times continuous (cake, pizza, the surface of a figure) and at times discrete (a set of balls or people). This unit is divided into "equal" parts, an adjective often not well defined in school, with often embarrassing results such as the following, concerning continuous situations:



or discrete ones: how to calculate $\frac{3}{5}$ of 12 people.

Providing students with concrete models and then requesting abstract reasoning, independently of the proposed model, is a clear indicator of a lack of didactic awareness on the part of the teacher and a sure recipe for failure.
2) At times a fraction is a quotient, a division not carried out, such as $\frac{a}{b}$, which should be interpreted as a:b; in this case the most intuitive interpretation is not that of part/whole, but that we have a objects and we divide them in b parts.
3) At times a fraction indicates a ratio, an interpretation which corresponds neither to part/whole nor to division, but is rather a relationship between sizes.
4) At times a fraction is an operator.
5) A fraction is an important part of work on probability, but it no longer corresponds to its original definition, at least in its ingenuous form.
6) In scores fractions have a quite different explanation and seem to follow a different arithmetic.
7) Sooner or later a fraction must be transformed into a rational number, a passage which is by no means without problems.
8) Later on a fraction must be positioned on a directed straight line, leading to a complete loss of its original sense

9) A fraction is often used as a measure, especially in its expression as a decimal number.
10) At times a fraction expresses a quantity of choice in a set, thereby acquiring a different meaning as an indicator of approximation.
11) It is often forgotten that a percentage is a fraction, again with particular characteristics.
12) In everyday language there are many uses of fractions, not necessarily made explicit, e.g. for telling the time ("A quarter to ten") or describing a slope (a 10% rise"), often far from a scholastic idea of fractions.

In this respect the studies of Vergnaud are illuminating. I am personally convinced that conceptual learning is the first stage of mathematical learning. So many different meanings for the concept of "fraction" require an attempt to find some unifying principle. Following Vergnaud, we can consider a *concept* C as three sets C = (S, I, *S*) such that:

- S is the set of situations that give sense to the concept (the referent);
- I is the set of the invariants on which is based the operativity of the schemata (the signified);
- *S* is the set of linguistic and non-linguistic forms that permit symbolic representation of the concept, its procedures, the situations and treatment procedures (the signifier).

Thus it is evident that the choice of a single meaning of fraction cannot conceptualize the fraction in its multiple features.
As we have seen:

- Behind the same term "fraction" are hidden may different situations which give sense to the concept
- Each of these situations contains invariants on which are based the operativity of the schemata,
- Various linguistic forma can be used to represent the concept.

Thus it is necessary to conceptualize the fraction via all of these meanings and not just through one or two of them, a scholastic choice that would lead to failure.

Vergnaud proposes also a theory of conceptual fields: "a set of situations, concepts and symbolic representations (signifiers) closely interdependent which cannot be analyzed separately" … "a set of problems and situations the handling of which requires different concepts, procedures and representations which are strictly interconnected".

On the one hand, it is impossible to imagine an approach to teaching fractions in isolation from the mathematical context which gives them sense: fractions, ratios, proportions, multiplications, rational numbers, are but a few of the emerging features from all of that which gives sense to fractions. These concepts must not be separated, but rather should flow together in one sole learning process. On the other, all the different acceptations of fractions must be explored and put in relationship between one another, since there are considerable differences between some of them.

Gérard Vergnaud's ternary schema is important and useful, as we have seen, but other approaches have been proposed for the conceptualization. More recently, Raymond Duval has replaced the ternary schema with a binary schema containing the pair "meaning-object" or "sign-object", thereby expressing the idea that conceptualization passes through the sign which expresses its own object. The occurrences of the mathematical object "fraction" are multiple and refer back to a variety of signs each one belonging to an appropriate system of signs.

6. The noetics and semiotics of fractions

The term "noetics" refers to conceptual acquisition and thus within the school environment to conceptual learning.
The term "semiotics" refers to the representation of concepts through systems of signs.
Both are of extraordinary importance in Mathematics.

On the one hand any form of mathematical activity requires the learning of its concepts. On the other it is impossible to study the learning in Mathematics without referring to semiotic systems.

It is important to bear in mind that the concepts of Mathematics do not exist in concrete reality. The point P, the number 3, addition, parallelism between straight lines, are not concrete objects which exist in empirical reality. They are pure concepts, ideal and abstract, and therefore , if we want to refer to them, they cannot be "empirically displayed" as in other sciences. In Mathematics concepts can only be *represented* by a chosen semiotic register.
As a matter of fact, in Mathematics we do not work directly with objects (i.e. with concepts), but with their *semiotic representations*. So semiotics, both in Mathematics and in Mathematics Education, is fundamental.

To represent a given concept there are many possible registers. Passing from one representation to another within the same register is called "transformation by treatment", while a change of semiotic representation into another register is called "transformation by conversion". In 1993 Duval called attention to a cognitive paradox hidden within these issues. We shall see that as regards the didactics of fractions this is an extraordinary important issue.

The objective is conceptual learning. The teacher (who knows the concept) proposes some of its semiotic representations to the student (who does not yet know the concept), in the hope, with the desire and will, that, via the semiotic representations, the student will be able to construct the desired conceptual learning (noetics) of the concept. But the student possesses only semiotic representations, objects (words, formulae, drawings, diagrams, etc.), but not the concept itself. If the student already knew the concept, he could *recognize it in those* semiotic representations, but since he does not know it, *he sees only* semiotic representations, i.e. concrete objects, ink marks on sheets of paper, chalk marks on a blackboard, etc.

The teacher who is unaware of noetics and semiotics may well cherish the illusion that, if the student manipulates the representations, then he is manipulating the concepts and thus the cognitive construction has taken place. In reality, it may well be that an incredible widespread ambiguity has arisen: the student has only learnt to manipulate the semiotic representations but has not at all constructed the concept and the teacher is suffering from an illusion.

In this respect, there are no miraculous recipes, there is only the need of awareness.
The teacher who is aware of this issue, cannot avoid focusing on the learning of his students, verifying if they really belong to the sphere of noetics and not only to the semiotic manipulation.

The fraction is a concept thus its learning is within noetics. As such, it cannot be concretely displayed. We can operate with a one-whole, an object, a cake, dividing it and obtaining a part. But the result is not the mathematical “fraction”, only the “fraction of that object”. Working with the semiotic register of concrete operations, we have shown a semiotic representation, not the concept.

We can use words to describe what we have done to the cake, thereby changing semiotic register and showing another semiotic representation, but not the concept.

We can pass to other examples, abstracting from the concrete object, the cake, going to another concrete object, for example a rectangle (better its area), but once again we have changed semiotic register thus providing another semiotic representation, not the concept.

At this point we normally go beyond the object (cake, rectangle, etc.) to its abstraction and the concept of fraction is supposed to have been constructed independently from the concrete model of departure considered as unit or whole. But often this is an illusion.

Up to this point the units are continuous objects: a cake or the surface of a rectangle. Passing to discrete units - e.g. 12 balls which must, however, be considered as one unit-whole - the register has been completely changed but it is taken for granted that the conversion spontaneously takes place. Indeed, at this point (maybe even before) an appropriate mathematical formalism is introduced, the written form of fractions, together with the terms "numerator" and "denominator": as a matter of fact a new semiotic representation in a different register is supplied, and thus a new kind of conversion.

…

All this, and more, normally takes place within a lesson of 30-40-50 minutes.

The distinguishing features that characterize the different objects are chosen– the act of dividing, the cake (continuous), the surface (continuous) of a rectangle, the set of balls (discrete), the formal writing with its specific names – treatment transformations (few) and conversion transformations (many) are continuously carried out, taking for granted that, if the student is capable of reproducing them, the teaching has been successful, the learning achieved and the concept constructed.

If, however, we recognize that semiotics and noetics are not the same thing and that learning to manipulate semiotic representations is not noetics, we can understand how, usually after apparent initial success, within a few lessons or within the following weeks or months students may be in grave crisis, having learnt to manipulate a few passages and registers, nothing more, but he has not at all constructed the concept we wanted him to construct.

7. Difficulties in the learning of fractions and Mathematics Education

Research has illustrated some errors which are typical in students the world over. Research has thoroughly and precisely studied and listed them. Below we summarize the most important.

1) Difficulty in ordering fractions and numbers written in the decimal notation.
2) Difficulty with operations between fractions and between rational numbers.
3) Difficulty in recognizing even the most common schemata.
4) Difficulty in handling the adjective “equal”.
5) Difficulty in handling equivalences.
6) Difficulty in handling the reduction to minimum terms.
7) Difficulty in handling non standard figures.
8) Difficulty in passing from a fraction to the unit that has generated it.
9) Difficulty in handling autonomously diagrams, figures or models.

Research has highlighted these typical errors and has classified them, but without using modern Mathematics Education considered as Learning Epistemology in the specific case of fractions, thereby turning over its point of view, using the results of the copious research that has been conducted over the past 40 years.
This need has pushed me to consider the main research topics into Mathematics Education and come back to the previous classical research into fractions under this point of view, looking for didactic and not mathematical motivations of these “typical errors” The following list considers but a few of the issues involved.

1. Didactic contract

(a) The “sum” of fractions: $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$ is not something the student proposes to the teacher because he believes it to be true, but because he thinks it may be acceptable by the teacher in terms of its form …
(b) In the context of a problem involving fractions, it is illusory to imagine that the student reasons, while choosing the appropriate operation to perform, when it is well known that, by contract, his objective is that of receiving a nod of approval and so is perfectly capable of producing a series of proposals often quite contradictory. The apparent absurdity (from the mathematical point of view) of the series of proposals gains a logic (from the point of view of didactics).

Many studies about the didactical contract have highlighted situations that implicitly were under everybody’s attention, without being clearly expressed and understood: students give up taking risks, abdicate the burden of responsibility for their own learning and act only in terms of the contract. With reference to fractions this is rather evident.

2. Excessive semiotic representations

Opening any textbook shows immediately the immense number of semiotic representations available for expressing fractions.

Handling these registers choosing the distinguishing features of the concept we must treat and convert, is not learnt *automatically.* This learning results from a process of explicit teaching in which the teacher must render the student co-responsible.

Teachers often underestimate this aspect, ignoring the warning of Duval and passing from one register to another, believing that the student follows. The teacher is able to jump from one register to another without problems, because he has already conceptualized: while in fact the student does not so, the student follows at the level of semiotic representatives, but not of meanings.

3. Prematurely formed images and models

Dealing with fractions, often an image can be transformed too quickly into a mental model, when it should still remain an image. Let us take some examples.

(a) The image of a unit-whole divided into *equal* parts, taking *equal* to be identity, congruency, superimposability, creates an effective and durable concept of fraction which then transforms into a model and has to be respected on all occasions and thus impedes the noetics of the fraction.
(b) The image of dividing a unit-whole in equal parts and taking some of them suggests semantically that this "some" cannot be "all". The model is easily reinforced, given that it coincides with a strong intuition, but then impedes the passage to a unit as a fraction $\frac{n}{n}$ and to improper fractions.
(c) The use of geometric figures is seen by students to be specific and meaningful, whereas for the adult it is random and generic. The continuous use of only rectangles or circles compels to a way of thinking in which an image, instead of being open, ductile and modifiable, becomes a persistent and stable model. If the fraction is proposed using different figures (triangle, trapezium,…) the student no longer grasps the noetics of the fraction because the situation is not a part of his model

4. Misconceptions

There are numerous examples of misconceptions concerning fractions. Many of the examples we have seen are ascribable to students' misconceptions which have then become premature models when instead they had to remain provisional images. We have seen examples. I recall misconceptions linked to order between fractions, based on that between natural numbers, to the simplification of fractions, to the handling of equivalences between fractions, to operations between fractions, to the choice of figures on which to operate with fractions …

5. Ontogenetic, didactic and epistemological obstacles

Many of the things to be learnt concerning fractions can be considered as true epistemological obstacles and are easily recognizable in the history and in the practice of teaching.
(a) The reduction of fractions to minimum terms has for long been a specific object of study in history, as is shown by the fact that the Egyptians who cultivated fractions for many centuries used only fractions with unitary numerators
(b) The passage from fractions to decimal numbers required over 4,500 years of Mathematics.
(c) The handling of zero in fractions has often created enormous problems in history, even for illustrious mathematicians.

6. Excess of didactic situations and lack of a-didactic situations

The situations that teachers propose for the learning of fractions are mainly didactic whereas they rarely correspond to a-didactic situations. The result is of almost total failure in the learning of fractions and rational numbers, and this is clearly pointed out in the international literature. Today we know that the construction of meaningful learning must pass through a-didactic situations, but that these are by no means the most used in teaching practice, while it should be so.

Bibliography

Adda J., Hahn C. (1995). Pourcentage et sens commun. In: Keitel C., Gellert U., Jablonka E., Müller M. (eds.) (1995). *Mathematics education and common sense*. Actes de la 47ème rencontre CIEAEM. Freie Universität Berlin. Berlino: Müller.

Adjiage R., Pluvinage F. (2000). Un registre géométrique unidimensionnel pour l'expression des rationnels. *Recherches en didactique des mathématiques*. 20, 1, 41-88.

Arrigo G., D'Amore B. (1992). *Infiniti*. Milano: Angeli.

Arrigo G., D'Amore B. (1999). "Lo vedo, ma non ci credo". Ostacoli epistemologici e didattici al processo di comprensione di un teorema di Georg Cantor che coinvolge l'infinito attuale. *L'insegnamento della matematica e delle scienze integrate*. 22B, 5, 465-494.

Arrigo G., D'Amore B. (2002). "Lo vedo ma non ci credo…", seconda parte. Ancora su ostacoli epistemologici e didattici al processo di comprensione di alcuni teoremi di Georg Cantor. *La matematica e la sua didattica*. 1, 4-57.

Arrigo G., D'Amore B., Sbaragli S. (in press). *Gli infiniti*. Bologna: Pitagora.

Bachelard G. (1938). *La formation de l'esprit scientifique*. Paris: Vrin.

Bagni G.T. (1996-'97). *Storia della matematica*. 3 volumes. Bologna: Pitagora.

Ball D. (1993). Halves, pieces and twoths: costructing and using representational contexts in teaching fractions. In: Carpenter T.P., Fennema E., Romberg T.A. (eds.) (1993). *Rational numbers: as integration of research*. Hilsdale (N.J.): Lawrence Erlbaum. 157-195.

Barbero R., Carignano I., Magnani R., Tremoloso G. (1996). Una ricerca sulle frazioni. *L'insegnamento della matematica e delle scienze integrate*. 19B, 351-376.

Basso M. (1991a). Un possibile itinerario didattico sulle frazioni nella scuola elementare. *L'insegnamento della matematica e delle scienze integrate*. 14, 7, 678-698.

Basso M. (1991b). Un possibile itinerario didattico sulle frazioni nella scuola elementare: classe quarta. *L'insegnamento della matematica e delle scienze integrate*. 14, 9, 877-897.

Basso M. (1992). Un possibile itinerario didattico sulle frazioni nella scuola elementare: classe quinta. *L'insegnamento della matematica e delle scienze integrate*. 15, 1, 73-96.

Behr M.J., Harel G., Post T., Lesh R. (1992). Rational number, ratio and proportion. In: Grouws D.A. (ed.) (1992). *Handbook of research on mathematics teaching and learning*. New York: Macmillan. 296-333.

Behr M.J., Harel G., Post T., Lesh R. (1993). Rational numbers: towards a semantic analysis-emphasis on the operator construct. In: Carpenter T.P., Fennema E., Romberg T.A. (eds.) (1993). *Rational numbers: as integration of research*. Hilsdale (N.J.): Lawrence Erlbaum. 13-47.

Behr M.J., Lesh R., Post T., Silver E.A. (1983). Rational-number concept. In: Lesh R.A., Landau M. (eds.) (1983). *Acquisition of mathematics concepts and processes*, New York: Academic Press. 92-126.

Behr M.J., Post T. (1988). Teaching rational number and decimal concepts. In: Post T. (ed.) (1980). *Teaching mathematics in grades k-8*. Boston: Allyn and Bacon. 190-231.

Behr M.J., Post T., Silver E., Mierkiewicz D. (1980). Theoretical foundations for instructional research on rational numbers. *Proceedings PME 1980*. 60-67.

Behr M.J., Post T., Wachsmuth I. (1986). Estimation and children's concept of rational number size. In: Shoen H.L., Zweng M.J. (eds.) (1986). *Estimation and mental computation. 1986 Yearbook*. Reston (Ma): NCTM.

Behr M.J., Wachsmuth I., Post T. (1985). Construct a sum: a measure of children's understanding of fraction size. *Journal for research in mathematics education*. 16, 120-131.

Bergen P.M. (1966). Action research on division of fractions. *Arithmetic teacher*. 13, 293-95.

Bezuk N.S., Bieck M. (1993). Current research on rational numbers and common fractions: Summary and implications for teachers. In. Douglas T.O. (ed.) (1993). *Research ideas for the classroom. Middle grades mathematics*. New York: MacMillan. 118-136.

Bidwell J.K. (1968). *A comparative study of the learning structures of three algorithms for the division of fractional numbers*. Doctoral thesis. University of Michigan.

Bohan H.J. (1970). *A study of the effectiveness of three learning sequences for equivalent fractions*. Doctoral thesis. University of Michigan.

Bonotto C. (1991). Numeri razionali. Approcci diversi e relative sperimentazioni didattiche. *L'insegnamento della matematica e delle scienze integrate*. 14, 7, 607-638.

Bonotto C. (1992). Uno studio sul concetto di numero decimale e di numero razionale. *L'insegnamento della matematica e delle scienze integrate*. 15, 5, 415-448.

Bonotto C. (1993). Origini concettuali di errori che si riscontrano nel confrontare numeri decimali e frazioni. *L'insegnamento della matematica e delle scienze integrate*. 16, 1, 9-45.

Bonotto C. (1995). Sull'integrazione delle strutture numeriche nela scuola dell'obbligo. *L'insegnamento della matematica e delle scienze integrate*. 18A, 4. 311-338.

Bonotto C. (1996). Sul modo di affrontare i numeri decimali nella scuola dell'obbligo. *L'insegnamento della matematica e delle scienze integrate*. 19A, 2, 107-132.

Bonotto C., Basso M. (1994). Analisi di indagini sui numeri decimali rivolti ad allievi ed insegnanti della scuola dell'obbligo. In: Basso M. et al. (eds.) (1994). *Numeri e proprietà*. Università di Parma. 93-98.

Bove D. et al. (1994). Indagine sulla conoscenza e le competenze al passaggio dalla scuola elementare alla media. Proposte di interventi. In. Basso et al. (eds.) (1994). *Numeri e proprietà*. University of Parma. 87-92.

Boyer C. (1968). *Storia della matematica*. Milano: Isedi. 1976. Original edition USA 1968.

Brousseau G. (1972a). Les processus de mathématisation. *Bulletin de l'association des professeurs de mathématique de l'enseignement public*. Numéro Spécial: *La Mathématique à l'école élémentaire*. (Text edited for the Acts of the Conference held in Clermont Ferrand in 1970).

Brousseau G. (1972b). Vers un enseignement des probabilités à l'école élémentaire. *Cahier de l'IREM de Bordeaux*. 11.

Brousseau G. (1974). L'enseignement des probabilités à l'école élémentaire. *Comptes Rendus de la XXVIe Rencontre de la CIEAEM*. Irem de Bordeaux.

Brousseau G. (1976). Les obstacles épistémologiques et les problèmes en mathématiques. *Comptes Rendus de la XXVIIIe Rencontre de la CIEAEM*. Louvain la Neuve. 101-117.

Brousseau G. (1980a). Les échecs electifs dans l'enseignements des mathématiques à l'école élémentaire. *Revue de laryngologie, otologie, rhinologie*. 101, 3-4, 107-131.

Brousseau G. (1980b). L'échec et le contrat. *Recherches en didactique des mathématiques*. 41, 177-182.

Brousseau G. (1980c). Problèmes de l'enseignements des décimaux. *Recherches en didactique des mathématiques*. 1, 1, 11-59.

Brousseau G. (1981). Problèmes de didactique des décimaux. *Recherches en didactique des mathématiques*. 2, 3, 37-127.

Brousseau G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en didactiques des mathématiques*. 4, 2, 165-198.

Brousseau G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en didactique des mathématiques*. 7, 2, 33-115.

Brown C.A. (1993). A critical analysis of teaching rational number. In: Carpenter T.P., Fennema E., Romberg T.A.. (eds.) (1993). *Rational numbers: as integration of research*. Hilsdale (N.J.): Lawrence Erlbaum. 197-218.

Bunt L.N.H., Jones P.S., Bedient J.D. (1987). *Le radici storiche delle matematiche elementari*. Bologna: Zanichelli.

Cannizzaro L. (1992). La prima educazione matematica nel settore aritmetico. *L'insegnamento della matematica e delle scienze integrate*. 15, 3, 236-248.

Carraher D.W., Dias Schliemann A.L. (1991). Children's understanding of fractions as expressions of relative magnitude. *Proceedings of PME XV*. Assisi. 184-191.

Carruccio E. (1958). *Matematica e logica nella storia e nel pensiero contemporaneo*. Torino: Gheroni.

Cassina U. (1961). *Dalla geometria egiziana alla matematica moderna*. Roma: Cremonese.

Castro E. (2001). Números decimales. In: Castro E. (ed.) (2001). *Didáctica de la Matemática en la educación primaria*. Madrid: Síntesis. 315-346.

Castro E., Torralbo M. (2001). Fracciones en el currículo de la educación primaria. In: Castro E. (ed.) (2001). *Didáctica de la Matemática en la educación primaria*. Madrid: Síntesis. 285-314.

Centeno J. (1988). *Los números decimales. ¿Por qué?, ¿Para qué?* Madrid: Síntesis.

Chace A. et al. (1927). *The Rind mathematical papyrus*. Oberlin (Ohio).

Chevallard Y. (1988). L'universe didactique et ses objects: fonctionnement et dysfonctionnement. *Interactions didactiques*. 9-37.

Chevallard Y. (1991). Dimension instrumentale, dimension sémiotique de l'acitivité mathématique. *Séminaire de Didactique des Mathématiques et de l'Informatique de Grenoble.* LSD2, IMAG, University J. Fourier, Grenoble.

Chevallard Y., Jullien M. (1989). *Sur l'enseignement des fractions au collège. Ingénierie, recherche, société*. Marsiglia: Publications de l'IREM.

Chiappini G., Pedemonte B., Molinari M. (2004). Le tecnologie didattiche nell'approccio ai numeri razionali. *L'insegnamento della matematica e delle scienze integrate*. 27 A-B, 479-512.

Cid E., Godino J.D., Batanero C. (2003). *Sistemas numéricos y su didáctica para maestros*. Granada: Facultad de Ciencias de la Educación.

Clements M.A., Del Campo G. (1990). How natural is fraction knowledge? In: Steffe L.P., Wood T. (eds.) (1990). *Trasforming children's mathematics education: international perspective*. Hillsdale (N.J.): Lawrence Erlbaum. 181-188.

Coburn T.G. (1973). *The effects of a ratio approach and a region approach on equivalent fractions and addition/substraction for pupils in grade four*. Doctoral thesis. University of Michigan.

Confrey J. (1994). Splitting, similarity and the rate of change: new approaches to multiplication and exponential function. In: Harel G., Confrey J. (eds.) (1994). *The development of multiplicative reasoning in the learning of mathematics*. State University of New York Press. 293-332.

Confrey J. (1995). Student voice in examining "splitting" as an approach to ratio, proportions and fractions. In: Meira L., Carraher D. (eds.) (1995). *Proceedings of the 19th PME*. Recife (Brasile). 1, 3-29.

Coxford A., Ellerbruch L. (1975). Fractional Number. In: Payne J.N. (ed.) (1975). *Mathematics learning in early chilhood*. Reston (Va): NCTM.

D'Amore B. (1991). logica Logica LOGICA, la didattica della logica fra gli 8 ed i 15 anni. In: D'Amore B. (ed.) (1991). *La Matematica fra gli 8 ed i 15 anni*. Bologna-Roma: Apeiron. 79-90.

D'Amore B. (1993). *Problemi. Pedagogia e psicologia della matematica in attività di problem solving*. Milano: Angeli. Preface by Gérard Vergnaud. [Published in Spanish, Madrid: Sintesis, 1997].

D'Amore B. (1998). Oggetti relazionali e diversi registri rappresentativi: difficoltà cognitive ed ostacoli – Relational objects and different representative registers: cognitive difficulties and obstacles. *L'educazione matematica*. 1, 7-28. [Artiche republished in Spanish: *Uno*. 15, 1998, 63-76].

D'Amore B. (1999). *Elementi di didattica della matematica*. Prefazione di Colette Laborde. Bologna: Pitagora. [Published in Spanish with prefaces by Colette Laborde and Ricardo Cantoral, 2006, Bogotà: Magisterio ed. In press in Portugese].

D'Amore B. (2001a). Concettualizzazione, registri di rappresentazioni semiotiche e noetica. *La matematica e la sua didattica*. 2, 150-173. [Artiche republished in various languages].

D'Amore B. (2001b). Un contributo al dibattito su concetti e oggetti matematici: la posizione "ingenua" in una teoria "realista" vs il modello "antropologico" in una teoria "pragmatica". *La matematica e la sua didattica*. 1, 4-30. [Artiche republished in various languages].

D'Amore B. (2001c). *Didattica della matematica*. Bologna: Pitagora.

D'Amore B. (2002a). Gérard Vergnaud. Item in *Enciclopedia Pedagogica*, Appendix A-Z. Brescia: La Scuola. 1508-1509.

D'Amore B. (2002b). Basta con le cianfrusaglie. *La vita scolastica*. 8, 14-18.

D'Amore B. (2003a). La complexité de la noétique en mathématiques ou les raisons de la dévolution manquée. *For the learning of mathematics*. 23, 1, 47-51.

D'Amore B. (2003b). *Le basi filosofiche, pedagogiche, epistemologiche e concettuali della didattica della matematica*. Prefazione di Guy Brousseau. Bologna: Pitagora. [Published in Spanish, with prefaces by Guy Brousseau and Ricardo Cantoral. 2005, México DF: Reverté. Published in Portugese, with prefaces by Guy Brousseau and Ubiratan D'Ambrosio. 2005, Sao Paolo: Escrituras].

D'Amore B. (2004). Il ruolo dell'Epistemologia nella formazione degli insegnanti di Matematica nella scuola secondaria. *La matematica e la sua didattica*. 4, 4-30.

D'Amore B. (2005). Pipe, cavalli, triangoli e significati. Contributo ad una teoria problematica del significato concettuale, da Frege e Magritte, ai giorni nostri. *L'insegnamento della matematica e delle scienze integrate*.

D'Amore B., Fandiño Pinilla M.I. (2002). Un acercamiento analítico al "triángulo de la didáctica". *Educación matemática*. México DF, México. 14, 1, 48-61.

D'Amore B., Fandiño Pinilla M.I. (2003). La formazione iniziale degli insegnanti di matematica in Italia. *La matematica e la sua didattica*. 4, 413-440.

D'Amore B., Fandiño Pinilla M.I. (2004). Cambi di convinzione in insegnanti di matematica di scuola secondaria superiore in formazione iniziale. *La matematica e la sua didattica*. 3, 27-50.

D'Amore B., Godino J.D., Arrigo G., Fandiño Pinilla M.I. (2003). *Competenze in matematica. Una sfida per il processo di insegnamento – apprendimento*. Bologna: Pitagora.

D'Amore B., Marazzani I. (2003). *Problemi di matematica nella scuola primaria*. Prefazione di Gérard Vergnaud. Bologna: Pitagora.

D'Amore B., Matteuzzi M. (1975). *Dal numero alla struttura*. Bologna: Zanichelli.

D'Amore B., Matteuzzi M. (1976). *Gli interessi matematici*. Venezia: Marsilio.

D'Amore B., Oliva P. (1994). *Numeri. Teoria, storia, curiosità, giochi e didattica nel mondo dei numeri*. Milano: Angeli.

D'Amore B., Sbaragli S. (2005). Analisi semantica e didattica dell'idea di "misconcezione": una proposta. *La matematica e la sua didattica*. 2.

D'Amore B., Speranza F. (eds.) (1989, 1992). *Lo sviluppo storico della matematica - Spunti didattici*. Due volumi: vol. I, 1989; vol II, 1992. Roma: Armando, Roma 1989.

D'Amore B., Speranza F. (eds.) (1995). *La matematica e la sua storia. Alcuni esempi per spunti didattici*. Milano: Angeli.

Davis G.E. (1989). Attainment of rational number knowledge. In: Ellerton N., Clements K. (eds.) (1989). *To challenge to change*. Victoria: Mathematical Ass. of Victoria.

Davis G.E., Hunting R.P. (1990). Spontaneous partitioning: preschoolers and discrete items. *Educational studies in mathematics*. 21, 367-374.

Davis G.E., Hunting R.P., Pearn C. (1993a). Iterates and relations: Elliot and Shannon's fraction schemes. In: Hirabayashi I., Nohda N., Shigematsu K., Lin K. (eds.) (1993). *Proceedings of the seventeenth conference of the international group of PME.* Vol. III. Tsukuba: Università di Tsukuba. 154-161.

Davis G.E., Hunting R.P., Pearn C. (1993b). What might a fraction mean to a child and how would a teacher know? *Journal of mathematical behaviour*. 12, 1, 63-76.

Davis R.B. (1988). Is percent un number? *The journal of mathematical behavior*. 7, 3, 299-302.

Desjardins M., Hetu J.C. (1974). Activités mathématiques dans l'enseignements des fractions. Montreal: University of Québec.

Dickson L., Brown M., Gibson O. (1984). *Children learning mathematics*. Londra: Cassell Education.

Douady R. (1984). *Jeux de cadres et dialectique outil-objet dans l'enseignement des mathématiques*. Tesi di stato, Università di Parigi. The same article appears in: *Recherches en didactique des mathématiques*. 7, 2, 1986, 5-31.

Dupuis C., Pluvinage F. (1981). La proportionnalité et son utilisation. *Recherches en didactique des mathématiques*. 2, 2, 165-212.

Duval R. (1988a). Ecarts sémantiques et cohérence mathématique. *Annales de didactique et de sciences cognitives*. 1, 7-25.

Duval R. (1988b). Approche cognitive des problèmes de géométrie en termes de congruence. *Annales de didactique et de sciences cognitives*. 1, 57-74.

Duval R. (1988c). Graphiques et équations. *Annales de didactique et de sciences cognitives*. 1, 235-253.

Duval R. (1993). Registres de représentations sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives,* ULP, IREM Strasbourg. 5, 37-65.

Duval R. (1995). *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels.* Berne: Peter Lang. [Also published in Spanish, printed in Colombia].

Duval R. (1996). Quel cognitif retenir en didactique des mathématiques? *Recherche en didactique des mathématiques,* 16, 3, 349-382.

Duval R. (1998). Signe et object (I). Trois grandes étapes dans la problématique des rapports entre répresentation et objet. *Annales de didactique et de sciences cognitives.* 6, 139-163.

Eisenlhor A. (1877). *Ein mathematisches handbuch der alten Aegypter*. Leipzig.

Ellerbruch L.W., Payne J.N. (1978). A teaching sequence from initial fraction concepts through the addition of unlike fractions. In: Suydam M.N., Rey R.E. (eds.) (1978). *Developing computational skills*. Reston (Va): NCTM. 129-147.

Erdos P., Strauss E. (1971). Representation of 1 by egiptian fractions. *American mathematical monthly*. 78, 302-303.

Fandiño Pinilla M.I. (2002). *Curricolo e valutazione in matematica.* Bologna: Pitagora.

Fandiño Pinilla M.I. (ed.) (2003). *Riflessioni sulla formazione iniziale degli insegnanti di matematica: una rassegna internazionale*. Bologna: Pitagora.
Fandiño Pinilla M.I. (2005). *Le frazioni, aspetti concettuali e didattici*. Bologna: Pitagora.
Fernández F. (2001). Proporcionalidad entre magnitudes. In: Castro E. (ed.) (2001). *Didáctica de la Matemática en la educación primaria*. Madrid: Síntesis. 533-558.
Figueras O. (1991). *Fractions in realistic mathematics education*. Dordrecht: Kluwer.
Figueras O., Filloy E., Valdemoros M. (1987). Some difficulties which obscure the appropriation of the fraction concept. *Proceedings of PME XI*. Montréal. Vol. I.
Fischbein E. (1985). Intuizione e pensiero analitico nell'educazione matematica. In: Chini Artusi L. (ed.) (1985). *Numeri e operazioni nella scuola di base*. Bologna: UMI-Zanichelli. 8-19.
Fischbein E. (1992). Intuizione e dimostrazione. In: Fischbein E., Vergnaud G. (1992). *Matematica a scuola: teorie ed esperienza*. Editor: B. D'Amore. Bologna: Pitagora. 1-24.
Gardiner A. (1997). *La civiltà egizia*. Torino: Einaudi.
Gardner M. (1979). Le curiose frazioni dell'antico Egitto danno luogo a rompicapi e a problemi di teoria dei numeri. *Le scienze*. 126, 100-102.
Galloway P.J. (1975). *Achievement and attitude of pupils toward initial fractional number concepts at various ages from six through ten years and of decimals at ages eight, nine and ten*. Doctoral thesis. University of Michigan. 21-49.
Giacardi L. (1993). La matematica nell'antico Egitto. In: Frosali A., Ottaviani N. (eds.) (1993). *Il pensiero matematico nella ricerca storica italiana*. Ancona: Irrsae Marche.
Giacardi L., Roero S.C.(1979). *La Matematica delle civiltà arcaiche – Egitto, Mesopotamia, Grecia*. Preface and introduction by Tullio Viola. Torino: Stampatori Didattica.
Gillings R. (1972). *The mathematics in the time of the pharaohs*. Cambridge: Mit.
Giménez J. (1986). Una aproximación didáctica a las fracciones egipcias. *Números*. 14, 57-62.
Giménez J. (1994). Del fraccionamento a las fracciones. *Uno*. 1, 101-118.
Giovannoni L. (1996). Misure di estensione superficiale nella scuola dell'infanzia. *La matematica e la sua didattica*. 4, 394-423.
Godino J.D. (1991). Hacia una teoría de la didáctica de las matemáticas. In: Gutierrez A. (ed.) (1991). *Area de conocimiento: didáctica de las matemáticas*. Madrid: Síntesis.
Godino J.D., Batanero C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en didactique des mathématiques,* 3, 325-355.
Graeber A.O., Tanenhaus E. (1993). Multiplication and division: from whole numbers to rational numbers. In: Douglas T.O. (ed.) (1993). *Research ideas for the classroom. Middle grades mathematics*. New York: MacMillan. 99-117.

Gray E.M. (1993). The transition from whole numbers to fraction. Relazione presentata all'incontro dell'International Study Group on the Rational Numbers of Arithmetic. Athens (Ga): University of Georgia.

Green G.A. (1969). *A comparison of two approaches, area and finding a part of, and two instructional materials, diagrams and manipulative aids on multiplication of numbers in grade five*. Doctoral thesis. University of Michigan.

Groff P. (1994). The future of fractions. *International journal of mathematics education in science and technology*. 25, 4.

Hahn C. (1999). Proportionnalité et pourcentage chez des apprentis vendeurs. Réflexion sur la rélation mathématiques / réalité dans une formation 'en alternance'. *Educational studies in mathematics*. 39, 229-249.

Hart K. (1980). From whole numbers to fractions and decimals. *Recherches en didactique des mathématiques*. 1, 1, 61-75.

Hart K. (1981). Fractions. In: Hart K. (ed.) (1981). *Children's understanding of mathematics*. 11-16. Londra: Murray. 68-81.

Hart K. (1985). Le frazioni sono difficili. In: Chini Artusi L. (ed.) (1985). *Numeri e operazioni nella scuola di base*. Bologna: Zanichelli. 144-166.

Hart K. (1988). Ratio and proportion. In: Behr M., Hiebert J. (eds.) (1988). *Number concepts and operations in the middle grades*. Reston (Va): NCTM. 198-219.

Hart K. (1989). Fractions: equivalence and addition. In: Johnson D.C. (1989). *Mathematical framework 8-13. A study of classroom teaching*. Windsor: Nfer-Nelson. 46-75.

Hart K., Sinkinson A. (1989). They're useful. Children's view of concrete materials. *Proceedings PME XIII*. Paris. Vol. II.

Hasemann K. (1979). *On the difficulties with fractions*. Osnabruck: Università di Osnabruck.

Hasemann K. (1981). On difficulties with fractions. *Educational studies in mathematics*. 12, 71-87.

Hunting R.P. (1984a). Learning fractions in discrete and continous quantity contexts'. In: Southwell B., Eyland R., Copper M., Collis K. (eds.) (1984). *Proceedings of the eighth international conference for the PME*. Sydney: Mathematical Ass. of New South Wales. 379-386.

Hunting R.P. (1984b). Understanding equivalent fractions. *Journal of science and mathematics education in S. E. Asia*. Vol. VII. 1, 26-33.

Hunting R.P. (1986). Rachel's schemes for constructing fractions knowledge. *Educational studies in mathematics*. 17, 49-66.

Hunting R.P., Davis G. (1991). *Recent research in psychology: early fraction learning*. New York: Springer-Verlag.

Hunting R.P., Davis G., Bigelow J.C. (1991). Higher order in young children's engagement with a fraction machine. In: Hunting R.P., Davis G. (eds.) (1991). *Recent research in psychology: early fraction learning*. New York: Springer-Verlag. 73-90.

Hunting R.P., Sharpley C.F. (1988) Fraction knowledge in preschool children. *Journal for research in mathematics education*. 19, 2, 175-180.

Hunting R.P., Pepper K.I., Gibson S.J. (1992). Preschooler's schemes for solving partitioning tasks. In: Greeslin W., Graham K. (eds.) (1992). *Proceedings of the sixteenth international conference of PME*. Durham: University of New Hampshire. 281-288.

Ifrah G. (1981). *Storia universale dei numeri*. Milano: A. Mondadori. 1983. Original edition France 1981.

Kamii C., Clark F.B. (1995). Equivalent fractions: their difficulty and educational implications. *Journal of mathematical behaviour*. 14, 365-378.

Kaput J.J., West M.W. (1994). Missing-value proportional reasoning problems: factors affecting informal reasoning patterns. In: Harel G., Confrey J. (eds.) (1994). *The development of multiplicative reasoning in the learning of mathematics*. State University of New York Press. 237-287.

Karplus R., Pulos S., Stage E.K. (1983). Proportional reasoning of early adolescents. In: Lesh R., Landau M. (eds.) (1983). *Acquisition of mathematics concepts and processes*. New York: Academic Press. 45-90.

Karplus R., Pulos S., Stage E.K. (1994). Early adolescents' proportional reasoning on 'rate' problems. *Educational studies in mathematics*. 14, 219-233.

Keijzer R., Buys K. (1996). Groter of kleiner. Een doorkijkie door een nieuwe leergang breuken. [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens*. 15, 3, 10-17.

Keijzer R., Terwel J. (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational studies in mathematics*. 47, 1, 53-73.

Kerslake D. (1986). *Fractions: children's strategies and errors*. Londra: Nfer-Nelson.

Kieren T.E. (1975). On the mathematical cognitive and instructional foundations of rational number. In: Lesh R.A. (ed.) (1976). *Number and measurement*. Columbus (Oh): Eric-Smeac. 101-144.

Kieren T.E. (1976). *Research on rational number learning*. Athens (Georgia).

Kieren T.E. (1980). The rational number construct – its elements and mechanisms'. In: Kieren T.E. (ed.) (1980). *Recent research on number learning*. Columbus: Eric/Smeac. 125-150.

Kieren T.E. (1983). Partitioning. Equivalence and the construction of rational number ideas. In: Zweng M., Green M.T., Kilpatrick J. (eds.) (1983). *Proceedings of IV ICME*. Boston. 506-508.

Kieren T.E. (1988). Personal knowledge of rational numbers. Its intuitive and formal development. In: Hiebert J., Behr M. (eds.) (1988). *Number concepts and operations in the middle grades*. Reston (Va): NCTM-Lawrence Erlbaum Ass. 162-181.

Kieren T.E. (1992). Rational and fractional numbers as mathematical and personal knowledge. Implications for curriculum and instruction. In: Leinhardt R., Putnam R., Hattrup R.A. (eds.) (1992). *Analysis of arithmetic for mathematics teaching*. Hillsdale (N.J.): Lawrence Erlbaum Ass.. 323-371.

Kieren T.E. (1993a). Bonuses of understanding mathematical understanding. Paper presented at the meeting of the International study group on Rational Numbers. Athens (Ga): University of Georgia.

Kieren T.E. (1993b). Rational and fractional numbers: from quotient fields to recursive understanding. In: Carpenter T.P., Fennema E., Romberg T.A. (eds.) (1993). *Rational numbers: an integration of the research*. Hillsdale (NJ): Lawrence Erlbaum Ass. 49-84.

Kieren T.E. (1993c). The learning of fractions: maturing in a fraction world. Paper presented at the meeting on the teaching and learning of fractions. Athens (Ga): University of Georgia.

Kieren T.E., Nelson D., Smith G. (1985). Graphical algorithms in partitioning tasks. *Journal of mathematical behaviour*. 4, 1, 25-36.

Kline M. (1971). *Storia del pensiero matematico*. 2 volumi. Torino: Einaudi. 1991. Original edition. USA 1971

Krich P. (1964). Meaningful vs. mechanical methods. *School science and mathematics*. 64, 697-708.

Lamon S.J. (1993). Ratio and proportion: connecting content and children’s thinking. *Journal for research in mathematics education*. 24, 1, 41-61.

Lancelotti P., Bartolini Bussi M.G. (eds.) (1983). *Le frazioni nei libri di testo. Contenuti teorici e strategie didattiche*. Rapporto Tecnico n. 2. Modena: Comune di Modena.

Leonard F., Sackur-Grisvard C. (1981). Sur deux règles implicites utilisées dans la comparaison des nombres décimaux positifs. *Bulletin APMEP*. 327, 47-60.

Llinares Ciscar S., Sánchez García M.V. (1988). *Fracciones*. Madrid: Síntesis.

Llinares Ciscar S. (2003). Fracciones, decimales y razon. Desde la relación parte-todo al razonamiento proporcional. In: Chamorro M.C. (ed.) (2003). *Didáctica de las Matemáticas*. Madrid: Pearson – Prentice Hall. 187-220.

Loria G. (1929). *Storia delle matematiche*. Milano: Hoepli.

Mack N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for research in mathematics education*. 21, 1, 16-33.

Mack N.K. (1993). Learning rational numbers with understanding: the case of informal knowledge. In: Carpenter T.P., Fennema E., Romberg T.A. (eds.) (1993). *Rational numbers: an integration of research*. Hillsdale (NJ): Lawrence Erlbaum As. 85-106.

Marcou A., Gagatsis A. (2002). Representations and learning of fractions. In: Rogerson A. (ed.) (2002). *The humanistic renaissance in mathematics education*. Palermo: Personal publication. 250-253.

Mariotti M.A., Sainati Nello M., Sciolis Marino M. (1995). Con quale idea di numero i ragazzi escono dalla scuola media? *L’insegnamento della matematica e delle scienze integrate*. 18A-18B, 5.

Minskaya G.I. (1975). Developing the concept of number by means of the relationship of quantities. *Soviet studies*. VII. 207-261.

Muangnapoe C. (1975). *An investigation of the learning of the initial concept and oral written symbols for fractional numbers in grades three and four*. Doctoral thesis. University of Michigan.

Nesher P., Peled I. (1986). Shifts in reasoning. The case of extending number concepts. *Educational studies in mathematics*. 17, 67-79.

Neugebauer O. (1957). *Le scienze esatte nell'antichità*. Milano: Feltrinelli. 1974. I ed. orig. USA 1957.

Neuman D. (1993). Early conceptions of fractions: a phenomenographic approach. In: Hirabayashi, Nohda N., Shigematsu K., Lin F. (eds.) (1993). *Proceedings of the seventeeth international conference for PME*. Vol. III. Tsukuba: University of Tsukuba. 170-177.

Noelting G. (1980). The development of proportional reasoning and the ratio concept. *Educational studies in mathematics*. 11. Part I: 217-253; Part II: 331-363.

Noss R. (1998). *Nuove culture, nuove Numeracy*. Bologna: Pitagora.

Novillis C.F. (1976). An analysis of the fraction concepts into a hierarchy of selected subconcepts and the testing of the hierarchical dependencies. *Journal of research in mathematics education*. 131-144.

Novillis C.F. (1980a). Seventh grade students' ability to associate proper fractions with points on the number line. In: Kieren T.E. (ed.) (1980). *Recent research on number learning*. Columbus (Oh): Eric-Smeac.

Novillis C.F. (1980b). Locating proper fractions on number line: effect on length and equivalence. *School science and mathematics*. LXXX, 5, 423-428.

O'Connor M.C. (2001). "Can any fraction be turned into a decimal?". A case study of a mathematical group discussion. *Educational studies in mathematics*. 46, 143-185.

Ohlsson S. (1988). Mathematical meaning and applicational meaning in the semantic of fractions and related concepts. In: Hiebert J., Behr M. (eds.) (1988). *Research agenda for mathematics education: number concepts and operations in the middle grades*. Reston (Va): NCTM-Lawrence Erlbaum Ass. 55-92.

Owens D. (1980). The relationship of area measurement and learning initial fractions concepts by children in grades three and four. In: Kieren T.E. (ed.) (1980). *Recent research on number learning*. Columbus (Oh): Eric-Smeac.

Payne J.N. (1975). Review of research on fractions. In: Lesh R.A. (ed.) (1975). *Number and measurement*. Columbus (Oh): Eric-Smeac.

Peano G. (1924). *Giochi di aritmetica e problemi interessanti*. Torino: Paravia. A more recent edition is: Firenze: Sansoni, 1983.

Pepper K.L. (1991). Preschoolers knowledge of counting and sharing in discrete quantity settings'. In: Hunting R.P., Davis G. (eds.) (1991). *Recent research in psychology: early fraction learning*. New York: Springer-Verlag. 103-127.

Peralta T. (1989). *Resolución de operaciones de suma y multiplicación de fracciones, en su forma algorítmica y su representación gráfica en los modelos continuo y discreto de fracción de la unidad*. Specialisation thesis, Department of Matemática Educativa. Cinvestav. México.

Piaget J., Inhelder B., Szeminska A. (1948). *La géometrie spontanée de l'enfant*. Paris: PUF. [Italian translation 1976, Firenze: Giunti-Barbèra].

Picutti E. (1977). *Sul numero e la sua storia*. Milano: Feltrinelli.
Pitkethly A., Hunting R. (1996). A review of recent research in the area of initial fraction concepts. *Educational studies in mathematics*. 30, 5-38.
Post T., Cramer K. (1987). Children's strategies in ordering rational numbers. *Arithmetic teacher*. 10, 33-36.
Pothier Y, Sawada D. (1983). Partitioning: the emergence of rational number ideas in youg children. *Journal for research in mathematics education*. 14, 5, 307-317.
Ratsimba-Rajohn H. (1982). Eléments d'étude de deux méthodes de mesure rationnelle. *Recherches en didactique des mathématiques*. 3, 1, 66-113.
Resnik L.B., Nesher P., Leonard F., Magone M., Omanson S., Peled I. (1989). Conceptual bases of arithmetic errors: the case of decimal fractions. *Journal for research in mathematics education*. 20, 1, 8-27.
Resnik L.B., Singer J.A. (1993). Protoquantitative origins of ratio reasoning. In: Carpenter P., Romberg T.A. (eds.) (1993). *Rational numbers: an integration of research*. Hillsdale (NJ): Lawrence Erlbaum. 107-130.
Rizzi B. (1988). Il calcolo frazionario egiziano. In 3 puntate. *Incontri con la matematica*. 3, 37-52; 4, 59-72; 5, 61-83. [Roma: Armando, editor B. D'Amore].
Roero C.S. (1987). *Numerazione e aritmetica nella matematica egizia*. Bari: Dedalo.
Rouchier A. et al. (1980). Situations et processus didactiques dans l'étude des nombres rationnels positifs. *Recherches en didactique des mathématiques*. 1, 2, 225-275.
Saenz-Ludlow A. (1990). A case study of the role of utilizing operations with natural numbers in the conceptualization of fractions. In: Boker G., Cobb P., De Mendicuti T.N. (eds.) (1990). *Procedings of the fourteenth international conference for the PME*. México D.F: Conacyt. 51-58.
Saenz-Ludlow A. (1992). Ann's strategie to add fractions. In: Geeslin W., Graham K. (eds.) (1992). *Proceedings of the sixteenth international conference fort he PME*. Vol. II. Durham (N.H.): University of New Hampshire. 266-273.
Saenz-Ludlow A. (1994). Michael's fraction schemes. *Journal for research in mathematics education*. 25, 50-85.
Saenz-Ludlow A. (1995). Ann's fraction schemes. *Educational studies in mathematics*. 28, 101-132.
Sbaragli S. (2003). Le convinzioni degli insegnanti elementari sull'infinito matematico. *L'insegnamento della matematica e delle scienze integrate*. Part I: 26A, 2, 155-186. Part II: 26A, 5, 573-588.
Sbaragli S. (2004). *Le convinzioni degli insegnanti sull'infinito matematico*. Doctoral thesis. University of Bratislawa. Published in Italian and English by GRIM of Palermo: http://math.unipa.it/~grim/tesi_it.htm
Sbaragli S. (2005). Misconcezioni "inevitabili" e misconcezioni "evitabili". *La matematica e la sua didattica*. 1.
Schubauer-Leoni M.L. (1996). Il contratto didattico come luogo di incontro, di insegnamento e di apprendimento. In Gallo E., Giacardi L., Roero C.S. (eds.)

(1996). *Conferenze e seminari 1995-1996*. Associazione Subalpina Mathesis - Seminario di Storia delle Matematiche “T. Viola”, Torino 17, 1, 7-27.

Sensevy G. (1994). Institutions didactiques, régulation, autonomie. Une étude des fractions au cours moyen. Doctoral thesis in Education. Aix-en-Provence.

Sensevy G. (1996a). Fabrication de problèmes de fraction par des élèves à la fin de l’enseignement élémentaire. *Educational studies in mathematics*. 30, 261-288.

Sensevy G. (1996b). Le temps didactique et la durée de l’élève. Étude d’un cas au cours moyen: le journal des fractions. *Recherches en didactique des mathématiques*. 16, 1, 7-46.

Sethe K. (1916). Von Zahlen und Zahlworten bei den alten Ägypten. *Schriften der Wissenschaftlichen Gesellschaft in Strassburg*. XXV.

Sfard A. (1991). On the dual nature of mathematical conceptions: reflection on processes and objects as differents sides of the same coin. *Educational studies in mathematics*. 22, 1-36.

Singh P. (2000). Understanding the concepts of proportion and ratio constructed by two grade six students. *Educational studies in mathematics*. 43, 271-292.

Sluser T.F. (1962). *A comparative study of division of fractions in which an explanation of the reciprocal principle is the experimental factor*. Doctoral thesis. University of Michigan.

Socas M. (2001). Problemas didácticos entre el objeto matemático y su representación semiótica. In: AA. VV. (2001). *Formación del profesorado e investigación en educación matemática*. Vol. III. La Laguna: University of La Laguna. 297-318.

Steffe L.P., Olive J. (1990). Constructing fractions in computer microworlds. In: Booker G., Cobb P., De Mendicuti T.N. (eds.) (1990). *Proceedings of the fourteenth international conference for the PME*. México DF: Conacyt. 59-66.

Steffe L.P., Parr R.B. (1968). *The development of the concepts of ratio and fraction in the fourth, fifth and sixth year of elementary school*. Washington D.C.: Department of Health, Education and Welfare.

Stenger D.J. (1971). An experimental comparison of two methods of teaching the addition and subtraction of common fractions in grade five. *Dissertation abstracts*. 32A. University of Cincinnati.

Streefland L. (1978). Subtracting fractions with different denominators. Some observational results concerning the mental constitution of the concept of fraction. *Educational studies in mathematics*. 9, 51-73.

Streefland L. (1979). Young children – ratio and proportion. *Educational studies in mathematics*. 10, 403-420.

Streefland L. (1982). Subtracting fractions with different denominators. *Educational studies in mathematics*. 13, 233-255.

Streefland L. (1983). Aanzet tot een nieuwe breukendidactiek volgens wiskobas. [Impuls for a new pedagogy on fractions according to wiskobas]. Utrecht: Owoc.

Streefland L. (1984a). *How to teach fractions and learning fractions*. Utrecht: Owoc.

Streefland L. (1984b). Basic principles for teaching and learning fractions. In: Bell M. et al. (eds.) (1984). *Theory, research and practice in mathematical education*. Nothingam: Shell Center for Mathematical Education.

Streefland L. (1984c). Search for the roots of ratio: some thoughts on the long term learning process (toward… a theory). *Educational studies in mathematics*. 15, 327-348; 16, 75-94.

Streefland L. (1987). Free production of fraction monographs. In: Bergeron J.C., Herscovics N., Kieran C. (eds.) (1987). *Proceedings of the XI PME*. Montreal. Vol. I. 405-410.

Streefland L. (1990). *Fractions in realistic mathematics education*. Dordrecht: Kluwer.

Streefland L. (1991). *Fractions in realistic mathematics education: a paradigm of developmental research*. Dordrecht: Kluwer.

Streefland L. (1993). Fractions: a realistic approach. In: Carpenter T.P., Fennema E., Romberg T.A. (eds.) (1993). *Rational numbers: an integration of research*. Hillsdale (NJ): Lawrence Erlbaum Ass. 289-325.

Struve W.W. (1930). Mathematischer Papyrus des Staatlichen Museum der Schönen Künste in Moskau. *Quellen und studien zur Geschichte der Mathematik*. Parte I. *Quellen*. 1, 30.

Suydam M.N. (1979). Review of recent research related to the concepts of fractions and the ratio. *Critical reviews in mathematics education*.

Tahan M. (1997). *L'uomo che sapeva contare. Le mille e una notte dei numeri*. Firenze: Salani.

Vaccaro V. (1998). Un mondo fantastico per le frazioni. *L'insegnamento della matematica e delle scienze integrate*. 21A, 4, 321-335.

Valdemoros M. (1992). *Análisis de los resultados obtenidos a través de un examen exploratorio del 'Lenguaje de las fraciones'*. Part I. México: Department of Matemática Educativa. Cinvestav.

Valdemoros M. (1993a). *La construcción del lenguaje de las fracciones y de los conceptos involucrados en él*. Doctoral thesis. Department of Matemática Educativa. Cinvestav, México.

Valdemoros M. (1993b). The language of fractions as un active vehicle for concepts. *Proceedings of XV Annual meeting of the North American Chapter of the PME*. I, 1233-1239.

Valdemoros M. (1993c). La construcción de significados a través de distintos sistemas simbólicos. *Memorias de lV Simposio internacional sobre investigación en Matemática Educativa*. 273-284.

Valdemoros M. (1994a). Various representations of the fraction throught a case study. *Proceedings of the XIX PME*. II, 16-23.

Valdemoros M. (1994b). Fracciones, referentes concretos y vínculos referenciales. *Memorias de la VIII Reunión Centroamericana y del Caribe sobre Formación de profesores e Investigación en Matemática Educativa*. 21-30.

Valdemoros M. (1997). Recursos intuitivos que favorecen la adición de fracciones. *Educación matemática*. 9, 3, 5-17.

Valdemoros M. (1998). La constancia de la unida en la suma de fracciones: un estudio de caso. In: Hitt F. (ed.) (1998). *Investigaciones en Matemática Educativa*. México: Grupo Editorial Iberoamerica. 465-481.

Valdemoros M. (2001). Las fracciones, sus referencias y los correspondientes significados de unidad: studio de casos. *Educación matemática*. 13, 1, 51-67.

Valdemoros M. (2004). Lenguaje, fracciones y reparto. *Relime*. 7, 3, 235-256.

Valdemoros M., Orendain M., Campa A., Hernández E. (1998). La interpretación ordinal de la fracción. In: Hitt F. (ed.) (1998). *Investigaciones en Matemática Educativa*. México: Grupo Editorial Iberoamerica. 441-455.

Van der Waerden B. (1954). *Science awakening*. Groningen: Noordhoff.

Van der Waerden B. (1980). The 2:n tale in the Rhind papyrus. *Centaurus*.

Vergnaud G. (1983). Multiplicative structures. In: Lesh R., Landau M. (eds.) (1983). *Acquisition of mathematics concepts and processes*. New York: Academic Press. 127-174.

Vergnaud G. (1984). Interactions sujet-situations. Comptes rendus 3ème école d'été en didactique des mathématiques. Orléans.

Vergnaud G. (1985a). Psicologia cognitiva ed evolutiva. Ricerca in didattica della matematica: alcune questioni teoriche e metodologiche. In: Chini Artusi L. (ed.) (1985). *Numeri e operazioni nella scuola di base*. Bologna: UMI-Zanichelli. 20-45.

Vergnaud G. (1985b). Il campo concettuale delle strutture moltiplicative e i numeri razionali. In: Chini Artusi L. (ed.) (1985). *Numeri e operazioni nella scuola di base*. Bologna: UMI-Zanichelli. 86-121.

Vergnaud G. (1988). Moltiplicative structures. In: Hiebert J., Behr M.J. (eds.) (1988). *Number concepts and operations in the middle grades*. Hillsdale (NJ): Lawrence Erlbaum Ass. 141-161.

Vergnaud G. (1990). La théorie des champs conceptuels. *Recherches en didactiques des mathématiques*. 10, 133-169.

Vergnaud G. (1992). Concetti e schemi in una teoria operatoria della rappresentazione. In: Fischbein E., Vergnaud G. (1992). *Matematica a scuola: teorie ed esperienze*. Editore B. D'Amore. Bologna: Pitagora. 103-124.

Vygotskij L.S. (1962). *Thought and language*. Cambridge, MIT Press.

Viola T. (1985). Le opinioni che gli antichi Greci avevano sulla matematica delle culture precedenti. *Scienza e filosofia. Saggi in onore di L. Geymonat*.

Vogel K. (1958). *Vorgriechische mathematic*. Vol. 1. *Vorgeschichte und Ägypten*. Hannover: Hermann Schroeder.

Wachsmuth I., Lesh R., Behr M. (1985). Order and equivalence of rational numbers: a cognitive analysis. *Journal of research in mathematics education*. 16, 18-36.

Weame D. (1990). Acquiring meaning for decimal fraction symbols: a one year follow-up. *Educational studies in mathematics*. 21, 545-564.

Weame D., Hiebert J. (1988). A cognitive approach to meaningful mathematics instruction: testing a local theory using decimal numbers. *Journal for research in mathematics education*. 19, 371-384.

Weame D., Kouba V.L. (2000). Rational numbers. In Silver E., Kenney P.A. (eds.) (2000). *Results from the 7th mathematics assessment of the national assessment of educational progress*. Reston (Va): NCTM. 163-192.

Williams H.B. (1975). *A sequential introduction of initial fraction concepts in grades two and four and remediation in grade six*. Research Report. University of Michigan: School of Education.

Wilson J.A. (1967). The effect of teaching the rationale of the reciprocal principle in the division of fractions through programmed instruction. *Dissertations abstracts*. 23A. University of Pittsburg.

Woodcock G.E. (1986). Estimating fractions: a picture is worth a thousand words. In: Shoen H.L., Zweng M.J. (eds.) (1986). *Estimation and mental computation*. Reston (Va): NCTM.

Zazkis R. (1998). Divisors and quotients: acknowledging polysemy. *For the learning of mathematics*. 18, 3, 27-29.